	3	4	6	3	5	4	
	5	2		2		3	
3	2		1	1	2		
6		3	1	1	3	2	4
	2	4	2		6	5	
		5		2	4	2	

Figure 1: Q1

Solution 1

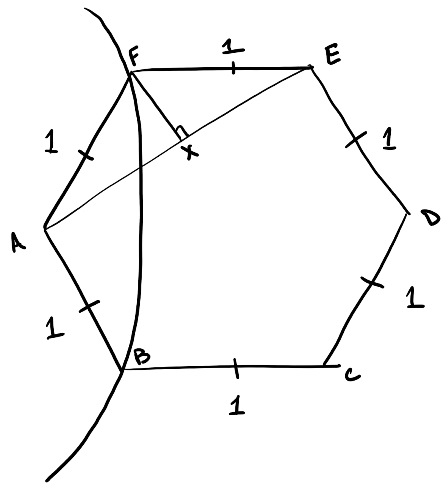

Figure 2: Case 1

Let the regular hexagon $ABCDEF$ with side length s be placed on the unit circle O where the vertices A coincides with the center of O.

We have to determine which two vertices lay on the circumference of the circle. Then using the radius we can find s. Since an equilateral hexagon is composed of 6 equilateral triangles, we can find the area using the formula: $6 * \frac{s^2 * \sqrt{3}}{4}$.

The two vertices P_1, P_2 that lay on the circumference of the circle must be equidistant from A, since AP_1 and AP_2 will be radii of the unit circle. We have two cases for P_1, P_2 :

Case 1: $P_1, P_2 = F, B$

Case 2: $P_1, P_2 = E, C$

Lemma 1: Points E, C lay on the circumference of the circle.

Proof of Lemma 1:

We can first show that case 1 does not create a valid configuration. If points F, B lay on the circumference of the circle, $s =$ radius of $O = 1$ The $\triangle\ AFE$ has side lengths $AF = FE = 1$,

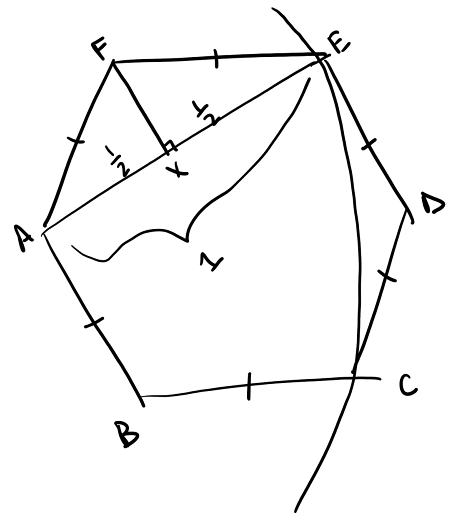

Figure 3: Case 2

$\angle AFE = 120$, and $\angle FAE = \angle FEA = 30$ Dropping the perpendicular from F to AE creates a perpendicular bisector of AE. Let this intersection between this perpendicular and AE be a point X . Refer to Figure 2

Since $\triangle$ AFX is a 30-60-90 triangle, $AX = \frac{\sqrt{3}}{2}$

Similarly, since $\triangle AFX \cong \triangle EFX$ by ASA, $AX = EX = \frac{\sqrt{3}}{2}$. By the segment addition postulate, $AE = AX + XE = \sqrt{3}$

$AE > AF$ so E must lay outside of the circle. By symmetry, C lays outside of the circle as well.

Thus, at least two points lay outside of the circle which means that this case does not satisfy the given conditions.

We can now show the case 2 is the correct arrangement. Refer to Figure 3 Let X be the intersection between the perpendicular dropped from F to AE We know $AX = XE = \frac{1}{2}$ since X is the perpendicular bisector of AE

$\triangle AFX$ is a 30-60-90 traingle so $AF = s = \frac{1}{\sqrt{3}}$

$AC = AE = 1 > \frac{1}{\sqrt{3}} = AF = AB$ so F and B must lay in the interior of the circle.

$\triangle AED$ is a right triangle so $AD = \frac{2}{\sqrt{3}}$ by the Pythagorean theorem. $AD > 1$, so D must lay outside of the unit circle. The given conditions are satisfied.

We now know that our arrangement is Case 2 and $AF = AB = s = \frac{1}{\sqrt{3}}$

Formula for area of equilateral triangle: $\frac{s^2 * \sqrt{3}}{4}$ where s is the side length of the triangle.

The area of a regular hexagon is composed of the areas of 6 identical equilateral triangles with side length s is:

$[ABCDEF] = \frac{6 * (\frac{1}{\sqrt{3}})^2 * \sqrt{3}}{4} = \boxed{\frac{\sqrt{3}}{2}}$

Claim: $M = 3$

Proof: We must show that for any distribution of the integers $1, 2, ..., 2024$ into groups A and B, there will be a fibtastic sequence of at least length 3 that is contained in either A or B. Thus, it must also be true that the minimum length of a fibtastic sequence in any distribution of 1 through 2024 is 3. Let the length of the largest fibtastic sequence in a distribution contained in either group be called the maxfib length. We can prove the claim is true by proving that a distribution cannot have a maxfib length of less than 3, then show that a maxfib length of 3 is valid. From this it follows that at a length 3 fibtastic sequence can be found in any distribution since either a distribution has a maxfib length of 3, or it contains fibtastic sequences of a longer length, which will contain fibtastic sequences of length 3.

Lemma 1: There cannot be a distribution where the maxfib length is less than 3.

Proof of Lemma 1: This can be proven via. proof by contradiction.

WLOG let $m >= n$ and assume this is a distribution where the maxfib length is less than 3. Let a_x and a_{x+1} be two consecutive elements in group A that are not in a fibtastic sequence with each other. The least difference a_x and a_{x+1} can have is 4, because it is the least positive integer not in the Fibonacci sequence. Since the sequence is strictly increasing, there will be at least 3 consecutive numbers in group B where $a_x < b_y, b_{y+1}, ...b_{y+a_{x+1}-a_x-2} < a_{x+1}$. Since a sequence of consecutive numbers all have a difference of 1, this sequence of at least length 3 in group B is fibtastic, which contradicts our initial assumption.

Lemma 2: A maxfib length of 3 is possible

Proof of Lemma 2:

$A : 1, 2, 3, 7, 8, 9, 13, 14, 15, 19, 20, 21, ...6n+1, 6n+2, 6n+3, ...2017, 2018, 2019, 2023, 2024$

$B : 4, 5, 6, 10, 11, 12, 16, 17, 18, 22, 23, 24, ...6n + 4, 6n + 5, 6n + 6, ...2020, 2021, 2022$

This distribution is constructed by placing 3 consecutive integers starting with 1, into group A. Then placing the next 3 consecutive integers, starting from 4, into group B. Then placing the next 3 consecutive integers into group A, placing the next 3 consecutive integers into group B and repeating this process until 2022 integers are placed. The remaining two integers, 2023 and 2024, can be placed in the group that is different from the group 2022 is placed in.

In this distribution there are 3 consecutive integers in each group, then a difference of 4 between the first term of the next 3 consecutive integers. This means that a fibtastic sequence of 3 is possible, but there can cannot be a fibtastic sequence of 4 since there are never are never more than 3 integers in a sequence without having a difference of 4 between a pair of consecutive terms. There are 2 remaining numbers after creating the groups using this method: 2023, 2024 which when placed at the end of group A do not create a longer fibtastic sequence than 3.

The sleeping periods of each mathematician can be represented on a line with 52 markings. Each marking represents a start or end time to a sleeping period, and since the time at which a mathematician falls asleep or wakes up is continuous, these markings will never be identical. We can index these markings $1, 2...52$ where an increase in the index represents a later time relative to the other times. Two indexes will be assigned to each mathematician with the smaller index representing the start of the mathematicians sleeping time. Let the start index of a mathematician be s_i and the end index e_i.

We can define the conditions that need to be met for a mathematician to not be sleeping at the same time as another mathematician. If a mathematician i has start index s_i and end index e_i, mathematician j must have either $e_j < s_i$ or $s_j > e_i$. There must be 6 of non intersecting blocks of sleep.

If two mathematicians are not not sleeping at the same time then they are sleeping at the same time, so we need to prove that the above conditions are true for 6 mathematicians, or they are false for 6 mathematicians pairwise for any selection of start points and endpoints.

We can rewrite $f(f(x)+x) < 0$ as $f(x)^2 + f(x)(2x+b+1) < 0$

Since $f(x)^2$ is always positive, $f(x)(2x+b+1)$ is negative so for the inequality to be true,

$$f(x)^2 < -f(x)(2x+b+1)$$

We now have two cases:

Case 1: $f(x) > 0$ and $(2x+b+1) < 0$

For $f(x)$ to be positive, x either must be less than both of it's roots, or it must be greater. than both of its roots.

The roots of $f(x)$ are $\frac{-b+\sqrt{b^2-4}}{2}$ and $\frac{-b-\sqrt{b^2-4}}{2}$ so,

$$x < \frac{-b-\sqrt{b^2-4}}{2} \text{ or } x > \frac{-b+\sqrt{b^2-4}}{2}$$

We can now find the remaining two roots.

$$f(x)^2 < -f(x)(2x+b+1)$$
$$f(x) < -(2x+b+1)$$
$$x^2+bx+1 < -(2x+b+1)$$
$$x^2+(b+2)x+(b+2) < 0$$

Since the quadratic must be negative, the valid x must fall in between the roots. So,

$$\frac{-b-2-\sqrt{b^2-4}}{2} < x < \frac{-b-2+\sqrt{b^2-4}}{2}$$

We can see that for case 1 there are two intervals in the domain for x: $(\frac{-b-2-\sqrt{b^2-4}}{2}, \frac{-b-\sqrt{b^2-4}}{2}), (\frac{-b-2+\sqrt{b^2-4}}{2}, \frac{-b+\sqrt{b^2-4}}{2})$ with each interval having a size of 1.

Case 2: $f(x) < 0$ and $(2x+b+1) > 0$

$$f(x)^2 < -f(x)(2x+b+1)$$
$$f(x) > (2x+b+1)$$

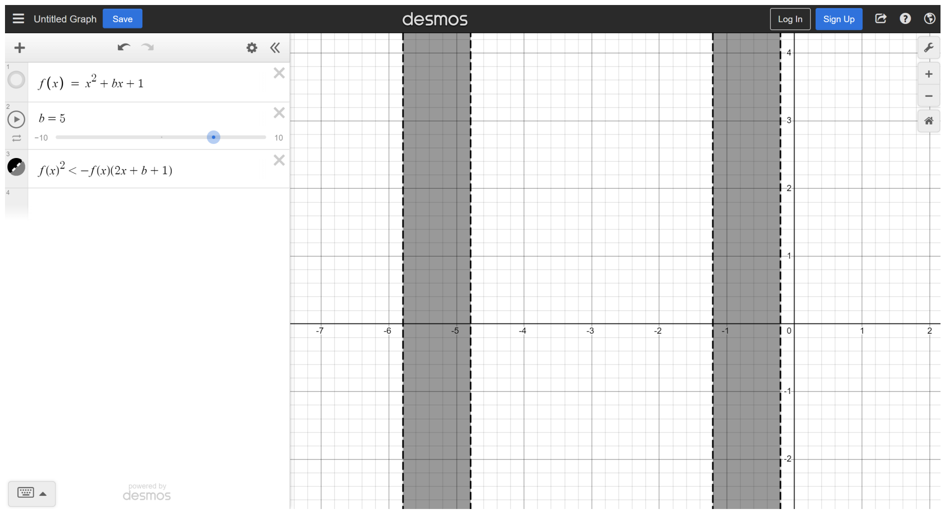

Figure 4: $b = 5$. Desmos graphic calculator with the following components: $f(x) = x^2 + bx + 1, b = 5, f(x)^2 < -f(x)(2x + b + 1)$

This is a contradiction because $f(x)$ is negative and $(2x + b + 1)$ is positive, so $f(x)$ cannot be less than $(2x + b + 1)$. Thus this case is not valid.

Now that we know Case 1 is the valid domain and that each of the intervals of the domain has size 1, we can find the different possibilities for the number of valid integer values of x and then show that each of these possibilities is true for some b

Since each interval is size 1, there cannot be two integer values in one interval and instead there is at most 1 in each interval. There is either 2 integer values (1 in each interval), 1 integer value(1 in either interval), or 0 integer values. However, for there to be no integer values of x in an interval, the boundaries of the interval must be integers. This is never possible because b must be an integer value but $\sqrt{b^2 - 4}$ is not an integer for integer values of b greater than 4 or less than -4.

We can show that the only possibility left, 2 integer values of x (1 in each interval) is valid by showing a b value where this is true.

When $b = 5$ the valid region shown in this graph (Figure 4), has 2 integer values of x.

Thus the only possible value for the number of integers that satisfy $f(f(x) + x) < 0$ is

$\boxed{2}$