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USA Mathematical Talent Search

Year	Round	Problem
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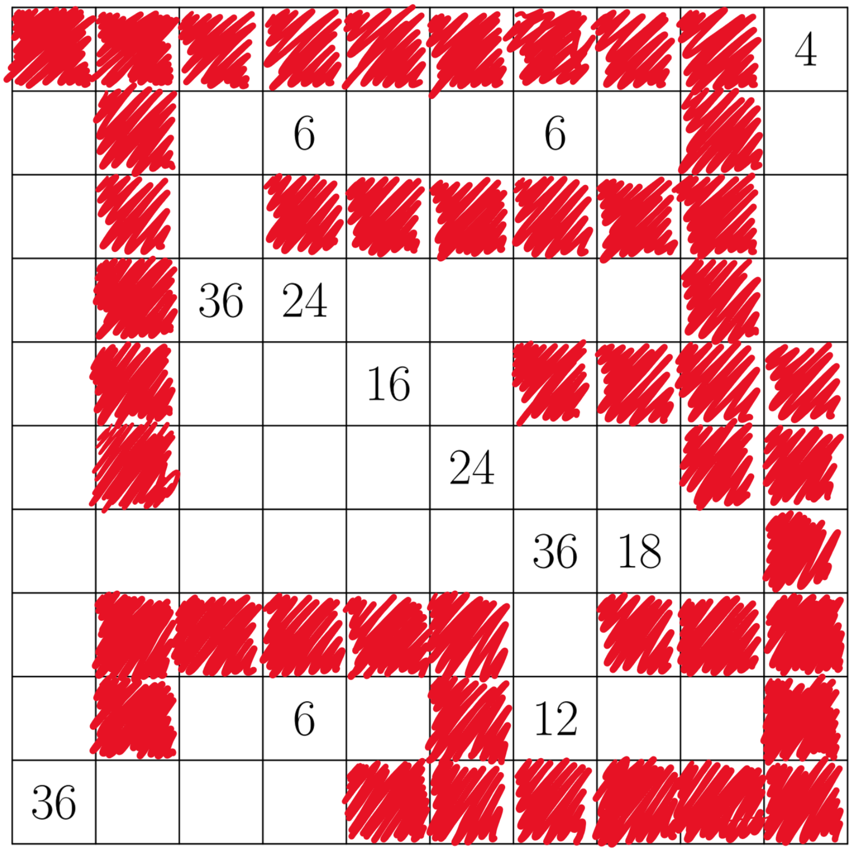

Figure 1: Enter Caption

Solution 1

The amount of rotation needed to set the clock to a valid time is determined by the position of the minute hand and the angle between the hour hand and the previous hour notch on the clock.

We must have both the hour and the minute hand represent the same time in minutes that have passed since the last top of the hour. For example, if the hour hand is pointing at 2 and the minute hand is pointing at 3, then as we rotate the hands, we will ensure that the minutes passed since 2:00 on the hour hand are the same as the minutes passed 2:00 on the minute hand. Since the minute hand starts at 15 minutes past 2, when it is rotated, its elapsed minutes will be added to 15.

Let the hour hand's position be h degrees clockwise from the closest hour notch to it, and let m be the minute hand position in degrees clockwise from the 12-notch. To ensure that all values are minimal, h and m will be taken modulo 360. Thus, when solving for the rotation, we will solve for the degrees clockwise modulo 360.

Since both the hour and minute hands must represent the same number of minutes passed since the previous hour, we can represent this value in terms of h, m, and x, where x is the clockwise rotation needed to create a valid time.

Each degree in the hour hand corresponds to 2 minutes since 30 degrees represent an hour. Thus, the minutes passed since the previous top of the hour in terms of h is $2h + 2x$. Each degree in the minute hand corresponds to $\frac{1}{6}$ minutes since there are 360 degrees in 60 minutes.

$$2(h + x) = \frac{m + x}{6}$$

Expanding and simplifying:

$$12h + 12x = m + x$$

$$x = \frac{m - 12h}{11}$$

The expected value for the hour hand is 15 degrees since the range is between 0 and 30 degrees, as it is placed between the notches uniformly. Similarly, the expected value of the minute hand is 180 degrees, since its range is between 0 and 360 degrees uniformly.

Plugging these values into the formula for x:

$$x = \frac{180 - 12(15)}{11} = \boxed{0}$$

The proof will be divided into two main cases based on the position of point P inside the quadrilateral. We will also consider a special case where P lies on one of the diagonals.

Case 1: Point P inside $\triangle ABC$

Assume that point P lies within $\triangle ABC$. We will use angle chasing to establish relationships between the angles in the triangles formed by point P.

Let:

$$\angle BAP = \angle CAD = x,$$
$$\angle BCP = \angle ACD = y.$$

Since $\overline{PB} \perp \overline{PD}$, the sum of angles in $\triangle APC$ must satisfy:

$$(45^\circ - 2x) + (135^\circ - 2y) + (x + y + \angle B) = 180^\circ.$$

Simplifying, we find:

$$x + y = \angle B.$$

This result shows that the angles x and y depend directly on the angle at vertex B. If $\overline{PB} \perp \overline{PD}$, it follows that $x + y = 45^\circ$, which implies that $\overline{AC} \perp \overline{BD}$.

Case 2: Point P inside $\triangle ACD$

Now, consider the case where point P lies within $\triangle ACD$. Using similar angle chasing techniques, we define:

$$\angle CAD = x,$$
$$\angle PAC = 45^\circ - 2x,$$
$$\angle PCA = 135^\circ - 2y.$$

The sum of angles in $\triangle APC$ again simplifies to:

$$x + y = \angle B.$$

Thus, we observe the same relationship as in Case 1, confirming that $\overline{PB} \perp \overline{PD}$ implies $\overline{AC} \perp \overline{BD}$ in this case as well.

Point P on $\overline{AC}$ or $\overline{BD}$

If point P lies directly on one of the diagonals, say $\overline{AC}$, then $\angle BAP = \angle CAD = 0°$. In this scenario, it is simple to verify the conditions manually:

$$\overline{PB} \perp \overline{PD} \implies \overline{AC} \perp \overline{BD}.$$

It is known that $\overline{AC} \perp \overline{BD}$, which implies that quadrilateral $ABCD$ is cyclic. In the context of triangle $\triangle ABC$, the circumcircle will be equidistant from each of the vertices A, B, and C, with the distance representing the radius of the circumcircle. This property extends to any combination of three vertices from the four points in the cyclic quadrilateral. Since the circumcircle of a triangle is uniquely determined by the circumcenter, which is equidistant from each vertex of the triangle, it follows that the circumcircle of any triangle formed by three of the four points will coincide with the circumcircle of the cyclic quadrilateral. If the distances $AP = BP = CP = DP$, the point P would be the circumcenter and, consequently, lie at the center of the circumcircle.

By applying angle chasing, we can derive results involving the circle and the four isosceles triangles, which allows us to demonstrate that$\overline{AC} \perp \overline{BD}$will always imply that $\overline{PB} \perp \overline{PD}$. A significant triangle in this context is $\triangle APC$, where $PAC = 45 - 2x = PCA = 135 - 2y$. By manipulating the relevant constants and variables, we obtain $y = 45 + x$.

Nevertheless, there remain several key angles that must be determined to prove $\overline{PB} \perp \overline{PD}$. To address this, we introduce the intersection point Q and define a new variable j as

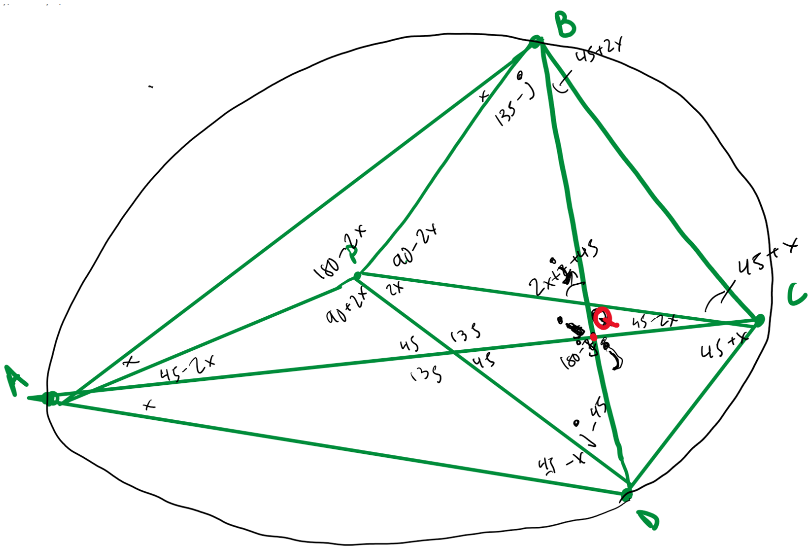

Figure 2: Angle Chase

the measure of $\angle AQB$.

By considering only the necessary angles for angle chasing, we can quickly deduce that $\angle PBD = 135° - x$ and $\angle PDB = j - 45°$. Since these angles are part of an isosceles triangle, it follows that $\overline{PQ} = \overline{PQ}$, confirming that $\overline{PB} \perp \overline{PD}$ if and only if $\overline{AC} \perp \overline{BD}$

As stated earlier, we also address the case where P lies on $\overline{AC}$. In this case, $\angle PAC = 0°$ and $\angle PCA = 0°$, which allows us to readily conclude that $\overline{PB} \perp \overline{PD}$ by further angle chasing.

Angle chasing show in diagram

$2^a * 5^b - 3^c = 1$

$2^a * 5^b = 1 + 3^c$

We can create cases for the coefficients in the equation.

Case 1: a and b are NOT 0 From examining the digits place we know that $c \equiv 2 \pmod 4$ The smallest solution for this case $(1, 1, 2)$.

We can create a recursive function for $2^a * 5^b$ $f_1 = 10$

$f_x = 81(f_{x-1} - 1) + 1$

We claim that $f_x \equiv 2 \pmod 4$. This can be proved by via induction.

Base case: $f_1 = 10 \equiv 2 \pmod 4$

Assume $f_x \equiv 2 \pmod 4$

$f_{x+1} = 1(2 - 1) + 1 \equiv 2 \pmod 4$

We claim that $f_x \equiv 0 \pmod 2$. This can be proved by induction.

$f_1 \equiv 0 \pmod 2$.

Assume $f_x \equiv 0 \pmod 2$

$f_{x+1} = 1(0 - 1) + 1 \equiv 0 \pmod 2$.

We now know that there is only one factor of 2 in $2^a * 5^b$ so $a = 1$ $2 * 5^b = 1 + 3^c$

We aim to prove that the equation

$$2 \cdot 5^b = 1 + 3^c,$$

where $b > 0$ and $c \geq 0$, has finitely many solutions using infinite descent.

We check small values of b and c:

For $b = 1$:

$$2 \cdot 5^1 - 1 = 10 - 1 = 9 \quad \Rightarrow \quad 3^c = 9 \quad \Rightarrow \quad c = 2.$$

Thus, $(b, c) = (1, 2)$ is a solution.

Thus, $(b, c) = (1, 2)$ are solutions for small values of b. We now proceed to show that no solutions exist for $b > 1$ using infinite descent.

Suppose (b, c) is a solution to the equation $2 \cdot 5^b - 1 = 3^c$ with $b > 1$ and $c > 2$. We aim to derive a smaller solution.

Since $5 \equiv 2 \mod 3$, we have $5^b \equiv 2^b \mod 3$. Therefore:

$$2 \cdot 5^b \equiv 2 \cdot 2^b \mod 3.$$

If b is odd, $2^b \equiv 2 \mod 3$, so $2 \cdot 5^b \equiv 1 \mod 3$, which implies:

$$1 + 3^c \equiv 1 \mod 3 \quad \Rightarrow \quad 3^c \equiv 0 \mod 3.$$

If b is even, $2^b \equiv 1 \mod 3$, so $2 \cdot 5^b \equiv 2 \mod 3$, which implies:

$$1 + 3^c \equiv 2 \mod 3 \quad \Rightarrow \quad 3^c \not\equiv 0 \mod 3.$$

This is a contradiction because 3^c must be divisible by 3. Hence, b must be odd.

Powers of $3 \mod 5$ cycle as $3, 4, 2, 1$. Since $5^b \equiv 0 \mod 5$, we have:

$$2 \cdot 5^b \equiv 0 \mod 5 \quad \Rightarrow \quad -1 \equiv 3^c \mod 5 \quad \Rightarrow \quad 3^c \equiv 4 \mod 5.$$

Thus, c must satisfy $3^c \equiv 4 \mod 5$, restricting c to specific values in the cycle.

Suppose $b > 1$. Rewriting the equation:

$$3^c = 2 \cdot 5^b - 1.$$

Let c' be the largest integer such that:

$$3^{c'} \leq 2 \cdot 5^b - 1.$$

Define the difference:

$$d = (2 \cdot 5^b - 1) - 3^{c'}.$$

Then:

$$d = 2 \cdot 5^b - 1 - 3^{c'},$$

where $0 < d < 3^{c'}$. This gives:

$$3^c = 3^{c'} + d.$$

Since $d < 3^{c'}$, we obtain a smaller instance of the original equation:

$$3^k = 2 \cdot 5^{b'} - 1,$$

where $b' < b$ and $k < c$.

This process of finding smaller solutions can be repeated indefinitely, reducing b and c at each step. Since b and c are non-negative integers, this infinite descent cannot continue indefinitely. Eventually, we reach $b = 1$, corresponding to the base case:

$$(b, c) = \quad (1, 2).$$

Case 2: $a = 0$ a cannot equal 0 because then $5^b - 3^c$ would equal an odd number which is not possible.

Case 3: $b = 0$

By examining the last digits of 2^a and 3^c we can see that the valid cases for the solutions modulo 4 are: $(1, 0, 0), (2, 0, 1), (3, 0, 3)$.

In all of these cases we have that $5^b = 1$ and $2^a - 1 = 3^c$

We aim to prove that the equation $2^a - 1 = 3^c$, where $a \geq 1$ and $c \geq 0$, has only finitely many solutions.

If $c = 0$, then

$$2^a - 1 = 1 \implies 2^a = 2 \implies a = 1.$$

Thus, $(a, c) = (1, 0)$ is a solution. If $c = 1$, then

$$2^a - 1 = 3 \implies 2^a = 4 \implies a = 2.$$

Thus, $(a, c) = (2, 1)$ is a solution. For $c \geq 2$, , we will use infinite descent to show that no solutions exist for $a > 2$ and $c > 1$.

Assume that (a, c) is a solution to the equation $2^a - 1 = 3^c$ with $a > 2$ and $c > 1$. We will show that this implies the existence of a smaller solution, eventually leading to a contradiction.

The equation can be rearranged as:

$$2^a = 3^c + 1.$$

If c is even, let $c = 2k$. Then:

$$3^c = (3^k)^2,$$

and the equation becomes:

$$2^a = (3^k - 1)(3^k + 1).$$

Since $3^k - 1$ and $3^k + 1$ are consecutive even numbers, their product 2^a implies that both factors must be powers of 2. Let:

$$3^k - 1 = 2^m \quad \text{and} \quad 3^k + 1 = 2^n,$$

where $m < n$ and $m + n = a$. Subtracting the two equations:

$$(3^k + 1) - (3^k - 1) = 2 \implies 2^n - 2^m = 2.$$

Factoring out 2^m:

$$2^m(2^{n-m} - 1) = 2.$$

This implies:

$$2^{n-m} - 1 = 1 \implies n - m = 1.$$

Thus:

$$n = m + 1 \quad \text{and} \quad a = m + n = 2m + 1.$$

Substituting back:

$$3^k - 1 = 2^m \quad \implies \quad 3^k = 2^m + 1.$$

This gives a smaller instance of the original equation:

$$2^m - 1 = 3^{k'},$$

where $m < a$ and $k' = k$. This provides a strictly smaller solution (m, k').

By repeating this process, we generate a strictly smaller solution at each step. Since a and c are positive integers, the descent cannot continue indefinitely. Eventually, we reach $a = 1$ or $c = 0$, corresponding to the base cases:

$$(a, c) = (1, 0) \quad \text{or} \quad (2, 1).$$

Using infinite descent, we have shown that the only solutions to the equation $2^a - 1 = 3^c$ are:

$$(a, c) = (1, 0) \text{ and } (2, 1).$$

Thus the only valid ordered solutions are:

$\boxed{(a,b,c) = (1,0,0), (2,0,1) \text{ and } (1,1,2)}$.